

# On the Erdős-Sós Conjecture for Graphs on $n = k + 4$ Vertices <sup>\*</sup>

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## Abstract

The Erdős-Sós Conjecture states that if  $G$  is a simple graph of order  $n$  with average degree more than  $k - 2$ , then  $G$  contains every tree of order  $k$ . In this paper, we prove that Erdős-Sós Conjecture is true for  $n = k + 4$ .

**Key words:** Erdős-Sós Conjecture; Tree; Maximum degree.

**AMS Classifications:** 05C05, 05C35.

## 1 Introduction

The graphs considered in this paper are finite, undirected, and simple (no loops or multiple edges). Let  $G = (V(G), E(G))$  be a simple graph of order  $n$ , where  $V(G)$  is the vertex set and  $E(G)$  is the edge set with size  $e(G)$ . The *degree* of  $v \in V(G)$ , the number of edges incident to  $v$ , is denoted  $d_G(v)$  and the set of neighbors of  $v$  is denoted  $N(v)$ . If  $u$  and  $v$  in  $V(G)$  are adjacent, we say that  $u$  *hits*  $v$  or  $v$  *hits*  $u$ . If  $u$  and  $v$  are not adjacent, we say that  $u$  *misses*  $v$  or  $v$  *misses*  $u$ . If  $S \subseteq V(G)$ , the

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induced subgraph of  $G$  by  $S$  is denoted by  $G[S]$ . Denote by  $D(G)$  the diameter of  $G$ . In addition,  $\delta(G)$ ,  $\Delta(G)$  and  $\text{avedeg}(G) = \frac{2e(H)}{|V(H)|}$  are denoted by the minimum, maximum and average degree in  $V(G)$ , respectively. Let  $T$  be a tree on  $k$  vertices. If there exists a injection  $g : V(T) \rightarrow V(G)$  such that  $g(u)g(v) \in E(G)$  if  $uv \in E(T)$  for  $u, v \in V(T)$ , we call  $g$  an *embedding* of  $T$  into  $G$  and  $G$  contains a copy of  $T$  as a subgraph, denoted by  $T \subseteq G$ . In addition, assume that  $T' \subset T$  is a proper subtree of  $T$  and  $g'$  is an embedding of  $T'$  into  $G$ . If there exists an embedding  $g : V(T) \rightarrow V(G)$  such that  $g(v) = g'(v)$  for all  $v \in V(T')$ , we say that  $g'$  is  *$T$ -extensible*.

In 1959, Erdős and Gallai [6] proved the following theorem.

**Theorem 1.1** *Let  $G$  be a simple graph with  $\text{avedeg}(G) > k - 2$ . Then  $G$  contains a path of order  $k$ .*

Based the above result, Later Erdős and Gallai proposed the following well known conjecture (for example see [7])

**Conjecture 1.2** *Let  $G$  be a simple graph with  $\text{avedeg}(G) > k - 2$ . Then  $G$  contains every tree on  $k$  vertices as a subgraph.*

Various specific cases of Conjecture 1.2 have already been proven. For example, Brandt and Dobson [2] proved the conjecture for graphs having girth at least 5. Balasubramanian and Dobson [1] proved this conjecture for graphs without containing  $K_{2,s}$ ,  $s < \frac{k}{12} + 1$ . Li, Liu and Wang [11] proved the conjecture for graphs whose complement has girth at least 5. In 2003, Mclellan [12] proved the following theorem.

**Theorem 1.3** *Let  $G$  be a simple graph with  $\text{avedeg}(G) > k - 2$ . Then  $G$  contains every tree of order  $k$  whose diameter does not excess 4 as a subgraph.*

In 2010, Eaton and Tiner [4] proved the the following two theorems.

**Theorem 1.4** [4] *Let  $G$  be a simple graph with  $\text{avedeg}(G) > k - 2$ . If  $\delta(G) \geq k - 4$ , then  $G$  contains every tree of order  $k$  as a subgraph.*

**Theorem 1.5** [4] *Let  $G$  be a simple graph with  $\text{avedeg}(G) > k - 2$ . If  $k \leq 8$ , then  $G$  contains every tree of order  $k$  as a subgraph.*

In 1984, Zhou [17] proved that Conjecture 1.2 holds for  $k = n$ . Later, Woźniak [16] proved that Conjecture 1.2 holds for  $k = n - 2$ .

**Theorem 1.6** [16] *Let  $G$  be a simple graph of order  $n$  with  $\text{avedeg}(G) > k - 2$ . If  $k = n - 2$ , then  $G$  contains every tree of order  $k$  as a subgraph.*

Recently, Tiner [15] proved that Conjecture 1.2 holds for  $k = n - 3$  holds.

**Theorem 1.7** [15] *Let  $G$  be a simple graph of order  $n$  with  $\text{avedeg}(G) > k - 2$ . If  $k \geq n - 3$ , then  $G$  contains every tree of order  $k$  as a subgraph.*

In this paper, we establish the following:

**Theorem 1.8** *Let  $G$  be a simple graph of order  $n$  with  $\text{avedeg}(G) > k - 2$ . If  $k \geq n - 4$ , then  $G$  contains every tree of order  $k$  as a subgraph.*

## 2 Proof of Theorem 1.8

Let  $T$  be any tree of order  $k$ . If  $k \geq n - 3$ , or  $k \leq 8$  or the diameter of  $T$  is at most 4, the assertion holds by Theorems 1.7, 1.5 and 1.3. We only consider  $k = n - 4 \geq 9$ ,  $D(T) \geq 5$  and prove the assertion by the induction. Clearly the assertion holds for  $n = 2$ . Hence assume Theorem 1.8 holds for all of the graphs of order fewer than  $n$  and let  $G$  be a graph of order  $n$ . If there exists a vertex  $v$  with  $d_G(v) < \lfloor \frac{k}{2} \rfloor$ , then  $\text{avedeg}(G - v) > k - 2$  and the assertion holds by the induction hypothesis. Further, by Theorem 1.4, without loss of generality, there exists a vertex  $z$  in  $V(G)$  such that  $\lfloor \frac{k}{2} \rfloor \leq d_G(z) = \delta(G) \leq k - 5$ . Moreover, assume that  $e(G) = 1 + \lfloor \frac{1}{2}(k - 2)(k + 4) \rfloor$ . Let  $T$  be any tree of order  $k$  with the longest path  $P = a_0 a_1 \dots a_{r-1} a_r$  and  $N_G(a_1) \setminus \{a_2\} = \{b_1, \dots, b_s\}$  and  $N_G(a_{r-1}) \setminus \{a_{r-2}\} = \{c_1, \dots, c_t\}$ . Since  $\text{avedeg}(G) > k - 2$ , we can consider the following cases:  $\Delta(G) = k + 3, k + 2, k + 1, k, k - 1$ .

### 2.1 $\Delta(G) = k + 3$

Let  $u \in V(G)$  be such vertex that  $d_G(u) = k + 3$  and let  $G' = G - \{u, z\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - (k + 3) - (k - 5) + 1 > \frac{1}{2}(k + 4)(k - 2) - (k + 3) - (k - 5) + 1 = \frac{1}{2}(k^2 - 2k - 2)$ . So  $\text{avedeg}(G') > (k^2 - 2k - 2)/(k + 2) > k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . Let  $f'$  be an embedding of  $T'$  into  $G'$ . Then let  $f = f'$  in  $T'$  and  $f(a_1) = u$ , where  $X = V(G) - f'(V(T'))$ . Since  $d_G(u) = k + 3$ ,  $u$  hits at least  $s$  vertices in  $X$ . Hence  $f$  can be extended to an embedding of  $T$  into  $G$  or we can say that  $f$  is  $T$ -extensible.

## 2.2 $\Delta(G) = k + 2$

Let  $u \in V(G)$  be such vertex that  $d_G(u) = k + 2$ . Then there exists only one vertex  $x \in V(G) - \{u\}$  not adjacent to  $u$ . We consider two subcases:  $d_G(x) \leq k - 2$  and  $d_G(x) \geq k - 1$ .

### 2.2.1 $d_G(x) \leq k - 2$

Let  $G' = G - \{u, x\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - (k + 2) - (k - 2) > \frac{1}{2}(k + 4)(k - 2) - (k + 2) - (k - 2) = \frac{1}{2}(k^2 - 2k - 8)$ . So  $avedeg(G') > (k^2 - 2k - 8)/(k + 2) = k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . Then let  $f'$  be an embedding of  $T'$  into  $G'$  and let  $f = f'$  in  $T'$  and  $f(a_1) = u$ , where  $X = V(G) - f'(V(T'))$ . Since  $d_G(u) = k + 2$ ,  $u$  hits at least  $s$  vertices in  $X$  and  $f$  is  $T$ -extensible.

### 2.2.2 $d_G(x) \geq k - 1$

We consider the following two cases.

(A).  $x$  misses  $z$ . Let  $G' = G - \{u, z, x\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_r\}$ . Then  $e(G') \geq e(G) - (k + 2) - (k - 5) - (k + 1) + 1 > \frac{1}{2}(k + 4)(k - 2) - (k + 2) - (k - 5) - (k + 1) + 1 = \frac{1}{2}(k^2 - 4k - 2)$ . Hence  $avedeg(G') > (k^2 - 4k - 2)/(k + 1) > k - 5$  and  $|V(T')| \leq k - 3$ . By the induction hypothesis, we have  $T' \subseteq G'$ . Since  $x$  misses  $z, u$  and  $d_G(x) \geq k - 1$ ,  $x$  misses at most two vertices of  $G'$ . If  $x$  hits  $f'(a_2)$ , let  $f(a_1) = x$  and  $f(a_r) = u$ . Since  $d_G(x) \geq k - 1$  and  $u$  hits all vertices of  $T'$ ,  $f$  is  $T$ -extensible. Hence we assume that  $x$  misses  $f'(a_2)$ . If  $x$  hits  $f'(a_{r-1})$ , let  $f(a_r) = x$  and  $f(a_1) = u$ . Then  $f$  is  $T$ -extensible. If  $x$  misses  $f'(a_2)$  and  $f'(a_{r-1})$ , then  $x$  hits all of  $V(G) - \{f'(a_2), f'(a_{r-1})\}$ , because  $D(T) \geq 5$ ,  $a_2$  and  $a_{r-1}$  are not adjacent. Then let  $f(a_{r-1}) = x, f(a_1) = u$ , which implies that  $f$  is  $T$ -extensible.

(B)  $x$  hits  $z$ . We consider the following two subcases.

(B.1)  $d_G(x) > k - 1$ . Let  $G' = G - \{u, z, x\}, T' = T - \{a_1, b_1, \dots, b_s, a_r\}$ . Since  $x$  misses  $u$  and  $d_G(x) > k - 1$ ,  $x$  misses at most two vertices of  $G'$ , the assertion can be proven by similar to method of (A).

(B.2).  $d_G(x) = k - 1$ . Then  $x$  misses 3 vertices of  $V(G) \setminus \{u\}$ , says  $y_1, y_2, y_3$ .

(a). There exists one vertex  $y_i$  with  $1 \leq i \leq 3$  such that  $d_G(y_i) = k + 2$ . Let  $G' = G - \{u, z, y_i, x\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 2) - (k - 5) - (k + 2) - (k - 1) + 3 + 1 > \frac{1}{2}(k + 4)(k - 2) - (k + 2) - (k - 5) - (k + 2) - (k - 1) + 3 + 1 = \frac{1}{2}(k^2 - 6k + 4)$ , because  $u$  hits  $z$ ,  $z$  hits  $x$ ,  $u$

hits  $y_i$ , and  $y_i$  hits  $z$  by  $d_G(y_i) = k + 2$ . Thus  $\text{avedeg}(G') > (k^2 - 6k + 4)/k > k - 6$  and  $|V(T')| \leq k - 4$ . By the induction hypothesis,  $T' \subseteq G'$ . Let  $f(a_1) = u$  and  $f(a_{r-1}) = y_i$ . Then  $f$  is  $T$ -extensible because  $u$  and  $y_i$  hits all the vertices of  $V(T')$ , respectively.

(b). There exists one vertex  $y_i$  with  $1 \leq i \leq 3$  such that  $d_G(y_i) = k + 1$ . Let  $G' = G - \{u, z, y_i, x\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 2) - (k - 5) - (k + 1) - (k - 1) + 3 > \frac{1}{2}(k + 4)(k - 2) - (k + 2) - (k - 5) - (k + 1) - (k - 1) + 3 = \frac{1}{2}(k^2 - 6k + 4)$ , which implies  $\text{avedeg}(G') > (k^2 - 6k + 4)/k > k - 6$  and  $|V(T')| \leq k - 4$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Note that  $y_i$  misses at most one vertex of  $G'$ . If  $y_i$  misses  $f'(a_2)$ , let  $f(a_1) = u, f(a_{r-1}) = y_i$ ; if  $y_i$  misses  $f'(a_{r-2})$ , let  $f(a_{r-1}) = u, f(a_1) = y_i$ . Thus  $f$  is  $T$ -extensible.

(c). There exists one vertex  $y_i$  with  $1 \leq i \leq 3$  such that  $d_G(y_i) = k$  and  $y_i$  misses  $z$ . Then the proof is similar to (b) and omitted.

(d). There exists one vertex  $y_i$  with  $1 \leq i \leq 3$  such that  $d_G(y_i) \leq k - 2$ . Let  $G' = G - \{u, y_i, x\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_r\}$ . Then  $e(G') \geq e(G) - (k + 2) - (k - 2) - (k - 1) + 1 > \frac{1}{2}(k + 4)(k - 2) - (k + 2) - (k - 2) - (k - 1) + 1 = \frac{1}{2}(k^2 - 4k - 4)$ , which implies  $\text{avedeg}(G') > (k^2 - 4k - 4)/(k + 1) > k - 5$  and  $|V(T')| \leq k - 3$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Similar to case (A), there exists an embedding from  $T$  into  $G$ .

(e).  $d_G(y_i) = k$  and  $y_i$  hits  $z$  for  $i \in \{1, 2, 3\}$ ; or  $d_G(y_i) = k - 1$  for  $i \in \{1, 2, 3\}$ .

(e.1)  $d_T(a_1) + d_T(a_{r-1}) \geq 5$ . Let  $G' = G - \{u, z, y_1, y_2, x\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 2) - (k - 5) - (k - 1) - (k - 1) - (k - 1) + 3 > \frac{1}{2}(k + 4)(k - 2) - (k + 2) - (k - 5) - (k - 1) - (k - 1) - (k - 1) + 3 = \frac{1}{2}(k^2 - 8k + 10)$  which implies  $\text{avedeg}(G') > (k^2 - 8k + 10)/(k - 1) > k - 7$  and  $|V(T')| \leq k - 5$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Moreover,  $x$  misses only one vertex of  $G'$ . If  $x$  misses  $f'(a_2)$ , let  $f(a_1) = u, f(a_{r-1}) = x$ ; if  $x$  misses  $f'(a_{r-2})$ , let  $f(a_{r-1}) = u, f(a_1) = x$ . In all situations,  $f$  is  $T$ -extensible.

(e.2).  $d_T(a_1) = d_T(a_{r-1}) = 2$ . Let  $G' = G - \{u, z\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - (k + 2) - (k - 5) + 1 > \frac{1}{2}(k + 4)(k - 2) - (k + 2) - (k - 5) + 1 = \frac{1}{2}(k^2 - 2k)$ , which implies  $\text{avedeg}(G') > (k^2 - 2k)/(k + 2) > k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . Moreover,  $u$  hits all vertices of  $V(G) \setminus \{x\}$  and  $z$  hits  $x$ . Let  $f(a_1) = u$  or  $z$  and  $f(a_0) = z$  or  $u$ . Then  $f$  is  $T$ -extensible.

### 2.3 $\Delta(G) = k + 1$

Let  $u \in V(G)$  be such vertex that  $d_G(u) = k + 1$  with  $u$  missing vertices  $x_1$  and  $x_2$ . Without loss of the generality, we can assume  $d_G(x_1) \geq d_G(x_2)$  and  $d_T(a_1) \geq d_T(a_{r-1})$ .

#### 2.3.1 $d_T(a_1) + d_T(a_{r-1}) \geq 5$

We consider the two cases.

(A).  $x_1$  misses  $x_2$ .

(A.1)  $d_G(x_1) + d_G(x_2) \leq 2k - 3$ . Let  $G' = G - \{u, x_1, x_2\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - (k + 1) - (2k - 3) > \frac{1}{2}(k + 4)(k - 2) - (k + 1) - (2k - 3) = \frac{1}{2}(k^2 - 4k - 4)$ , which implies  $avedeg(G') > (k^2 - 4k - 4)/k + 1 > k - 5$  and  $|V(T')| \leq k - 3$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Let  $f(a_1) = u$ . It is easy to see that  $f$  is  $T$ -extensible.

(A.2).  $d_G(x_1) + d_G(x_2) \geq 2k - 2$ .

(a).  $d_G(x_1) = k - 1$  Then  $d_G(x_2) = k - 1$  and  $x_1$  misses  $\{u, x_2, y_1, y_2\}$ . If  $y_1, y_2 \neq z$ , let  $G' = G - \{u, z, x_1, x_2, y_1\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 1) - (k - 5) - (2k - 2) - (k + 1) + 3 > \frac{1}{2}(k^2 - 8k + 8)$ , which implies  $avedeg(G') > (k^2 - 8k + 8)/(k - 1) > k - 7$  and  $|V(T')| \leq k - 5$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Note that  $x_1$  misses only one vertex of  $G'$ . If  $x_1$  misses  $f'(a_2)$ , let  $f(a_1) = u$  and  $f(a_{r-1}) = x_1$ ; if  $x_1$  misses  $f'(a_{r-1})$ , let  $f(a_{r-1}) = u$  and  $f(a_1) = x_1$ . In both situations,  $f$  is  $T$ -extensible. Now assume that  $y_1 = z$  or  $y_2 = z$ . Let  $G' = G - \{u, x_1, x_2, y_1, y_2\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 1) - (k - 5) - (2k - 2) - (k + 1) + 2 + 1 > \frac{1}{2}(k^2 - 8k + 8)$ , which implies  $avedeg(G') > (k^2 - 8k + 8)/(k - 1) > k - 7$  and  $|V(T')| \leq k - 5$ . Let  $f(a_{r-1}) = u$  and  $f(a_1) = x_1$ . Then  $f$  is  $T$ -extensible.

(b).  $d_G(x_1) \geq k$ . Let  $G' = G - \{u, z, x_1, x_2\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 1) - (k - 5) - (2k + 2) + 1 + 2 > \frac{1}{2}(k^2 - 6k + 2)$ , which implies  $avedeg(G') > (k^2 - 6k + 2)/k > k - 6$  and  $|V(T')| \leq k - 4$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Note that  $x_1$  misses at most one vertex of  $G'$ . If  $x_1$  misses  $f'(a_2)$ , let  $f(a_1) = u$  and  $f(a_{r-1}) = x_1$ ; if  $x_1$  misses  $f'(a_{r-2})$ , let  $f(a_{r-1}) = u$  and  $f(a_1) = x_1$ . In both situations,  $f$  is  $T$ -extensible.

(B).  $x_1$  hits  $x_2$ .

(B.1).  $d_G(x_1) + d_G(x_2) \leq 2k - 2$ . Let  $G' = G - \{u, x_1, x_2\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - (k + 1) - (2k - 2) + 1 > \frac{1}{2}(k^2 - 4k - 4)$ , which implies  $avedeg(G') > (k^2 - 4k - 4)/(k + 1) > k - 5$  and  $|V(T')| \leq k - 3$ . Hence

by the induction hypothesis,  $T' \subseteq G'$ . Let  $f(a_1) = u$ . It is easy to see that  $f$  is  $T$ -extensible.

(B.2).  $d_G(x_1) + d_G(x_2) \geq 2k - 1$ .

(a).  $d_G(x_1) = k$  Then  $d_G(x_2) = k - 1$  or  $k$ , and  $x_1$  misses  $u, y_1, y_2$ . If  $z \neq y_1, y_2$ , then let  $G' = G - \{u, z, x_1, x_2, y_1\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 1) - (k - 5) - 2k - (k + 1) + 4 + 1 > \frac{1}{2}(k^2 - 8k + 8)$ , which implies  $avedeg(G') > (k^2 - 8k + 8)/(k - 1) > k - 7$  and  $|V(T')| \leq k - 5$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Note that  $x_1$  misses only one vertex of  $G'$ . If  $x_1$  misses  $f'(a_2)$ , let  $f(a_1) = u$  and  $f(a_{r-1}) = x_1$ ; if  $x_1$  misses  $f'(a_{r-2})$ , let  $f(a_{r-1}) = u$  and  $f(a_1) = x_1$ . In both situations,  $f$  is  $T$ -extensible. If  $z = y_1$  or  $y_2$ , then let  $G' = G - \{u, x_1, x_2, z\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 1) - 2k - (k - 5) + 2 > \frac{1}{2}(k^2 - 6k + 4)$ , which implies  $avedeg(G') > (k^2 - 6k + 4)/k > k - 6$  and  $|V(T')| \leq k - 4$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Note that  $x_1$  misses only one vertex of  $G'$ . It is easy to see that there exists an  $f$  such that  $f$  is  $T$ -extensible.

(b).  $d_G(x_1) = k + 1$ . Let  $G' = G - \{u, x_1, x_2, z\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - (k + 1) - (k - 5) - (2k + 2) + 2 > \frac{1}{2}(k^2 - 6k)$ , which implies  $avedeg(G') > (k^2 - 6k)/k = k - 6$  and  $|V(T')| \leq k - 4$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Note that  $x_1$  misses at most one vertex of  $G'$ . It is easy to find an embedding of  $T$  into  $G$ .

### 2.3.2 $d_T(a_1) = d_T(a_{r-1}) = 2$ .

(A). There exists a vertex  $v \neq u$  of degree at most  $k$  such that it hits both  $x_1$  and  $x_2$ . Let  $G' = G - \{u, v\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - (k + 1) - k + 1 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $avedeg(G') > (k^2 - 2k - 8)/(k + 2) = k - 4$  and  $|V(T')| \leq k - 2$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2)$  hits  $u$ , then  $f(a_1) = u$ . If  $f'(a_2)$  misses  $u$ , then  $f'(a_2) = x_1$  or  $x_2$  and let  $f(a_1) = v, f(a_0) = u$ . Thus  $f$  is  $T$ -extensible.

(B). There exists a vertex  $v \neq u$  of degree at least  $k + 1$  such that it hits both  $x_1$  and  $x_2$ . Then  $d_G(v) = k + 1$  and  $v$  misses  $y_1$  and  $y_2$ . Let  $G' = G - \{u, v, z\} - \{x_1 x_2, y_1 y_2\}$  and  $T' = T - \{a_0, a_1, a_r\}$ . Then  $e(G') \geq e(G) - 2(k + 1) - (k - 5) + 1 - 2 > \frac{1}{2}(k^2 - 4k - 4)$ , which implies  $avedeg(G') > (k^2 - 4k - 4)/(k + 1) > k - 5$  and  $|V(T')| \leq k - 3$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_1$  or  $x_2$ , and  $f'(a_{r-1}) = y_1$  or  $y_2$ , then let  $f(a_1) = v$  and  $f(a_r) = u$ . If  $f'(a_2) = x_1$  and  $f'(a_{r-1}) = x_2$ , then let

$f(a_1) = v, f'(a_{r-1}) = u$ , because  $u$  hits all the neighbours of  $f'(a_{r-1})$ . If  $f'(a_2) = y_1, f'(a_{r-1}) = y_2$ , then let  $f(a_1) = u$  and  $f'(a_{r-1}) = v$ . For the rest situations, it is easy to find an embedding from  $T$  into  $G$ .

(C). There are no vertices in  $V(G) \setminus \{u\}$  hitting both  $x_1$  and  $x_2$ , and  $x_1$  misses  $x_2$ . Then  $d_G(x_1) + d_G(x_2) \leq k + 1$ . Let  $G' = G - \{u, x_1, x_2\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - (k + 1) - (k + 1) > \frac{1}{2}(k^2 - 2k - 12)$ , which implies  $avedeg(G') > (k^2 - 2k - 12)/(k + 1) > k - 4$  and  $|V(T')| \leq k - 2$ . By theorem 1.7,  $T' \subseteq G'$ . Let  $f(a_1) = u$ . Then  $f$  is  $T$ -extensible.

(D). There are no vertices in  $V(G) \setminus \{u\}$  hitting both  $x_1$  and  $x_2$ , and  $x_1$  hits  $x_2$ . Then  $d_G(x_1) + d_G(x_2) \leq k + 3$ . If  $d_G(x_1) + d_G(x_2) \leq k + 2$ , the assertion follows from (C). Hence assume that  $d_G(x_1) + d_G(x_2) = k + 3$ . Then  $z$  has to hit  $x_1$  or  $x_2$ , say that  $z$  hits  $x_1$ . Let  $G' = G - \{u, z\} - \{x_1, x_2\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - (k + 1) - (k - 5) + 1 - 1 > \frac{1}{2}(k^2 - 2k)$ , which implies  $avedeg(G') > (k^2 - 2k)/(k + 2) > k - 4$  and  $|V(T')| \leq k - 2$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2)$  hits  $u$ , let  $f(a_1) = u$ ; if  $f'(a_2) = x_1$ , let  $f(a_1) = z$  and  $f(a_0) = u$ . If  $f'(a_2) = x_2$  and if there is a vertex  $w$  in  $T'$  such that  $f'(w) = x_1$ , let  $f'(w) = u$ ,  $f(a_1) = x_1$  and  $f(a_0) = z$ , because  $u$  hits all neighbours of  $f'(w)$ ; if  $f'(a_2) = x_2$  and there does not exist any vertex  $w$  in  $T'$  such that  $f'(w) = x_1$ , let  $f(a_1) = x_1$ , and  $f(a_0) = z$ . In all situations,  $f$  is  $T$ -extensible.

## 2.4 $\Delta(G) = k$

Let  $u \in V(G)$  be a vertex of degree  $d_G(u) = k$  and miss three vertices  $x_1, x_2, x_3$ . Denote by  $S = \{x_1, x_2, x_3\}$

### 2.4.1 $G[S]$ contains no edges.

Let  $G' = G - \{u\}$  and  $T' = T - \{a_0\}$ . Then  $e(G') \geq e(G) - k > \frac{1}{2}(k^2 - 8)$ , which implies  $avedeg(G') > (k^2 - 8)/(k + 3) > k - 3$  and  $|V(T')| \leq k - 1$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_1)$  hits  $u$ , let  $f(a_0) = u$ ; if  $f'(a_1) = x_i$ ,  $1 \leq i \leq 3$ , let  $f'(a_1) = u$ . Since  $u$  hits all neighbours of  $f'(a_1)$ ,  $f$  is  $T$ -extensible.

### 2.4.2 $G[S]$ contains exactly one edge.

Without loss of the generality,  $x_1$  hits  $x_2$ . We consider two cases.

(A).  $d_T(a_1) + d_T(a_{r-1}) \geq 5$ .



(A.1).  $d_G(x_1) \geq k-1$  and  $d_G(x_2) \geq k-1$ . If  $x_3 \neq z$ , let  $G' = G - \{u, z, x_3\} - \{x_1x_2\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - k - (k-5) - k - 1 > \frac{1}{2}(k^2 - 4k)$ , which implies  $\text{avedeg}(G') > (k^2 - 4k)/(k+1) > k-5$  and  $|V(T')| \leq k-3$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2)$  hits  $u$ , then let  $f(a_1) = u$ ; if  $f'(a_2) = x_1$  and  $x_2 \notin f'(V(T'))$ , then let  $f(a_1) = x_2$ ; if  $f'(a_2) = x_1$  and  $x_2 \in f'(V(T'))$  and  $f'(w) = x_2$ , then let  $f'(w) = u$ ,  $f(a_2) = x_1$ , and  $f(a_1) = x_2$ . Hence  $f$  is  $T$ -extensible. On the other hand, if  $x_3 = z$ , let  $G' = G - \{u, z\} - \{x_1x_2\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Similarly, we can prove that the assertion holds.

(A.2).  $d_G(x_3) \geq k-1$ . By (A.1), we can assume that  $d_G(x_1) \leq k-2$  or  $d_G(x_2) \leq k-2$ , say  $d_G(x_1) \leq k-2$ . If  $z \neq x_1, x_2$ , let  $G' = G - \{u, z, x_1, x_2, x_3\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - k - (k-5) - (k-2) - k - k + 2 + 1 > \frac{1}{2}(k^2 - 8k + 12)$ , which implies  $\text{avedeg}(G') > (k^2 - 8k + 12)/(k-1) > k-7$  and  $|V(T')| \leq k-5$ . Hence by the induction hypothesis,  $T' \subseteq G'$ . Moreover,  $x_3$  misses at most one vertex of  $V(G')$ . If  $x_3$  misses  $f'(a_2)$ , let  $f(a_1) = u$  and  $f(a_{r-1}) = x_3$ ; if  $x_3$  hits  $f'(a_2)$ , let  $f(a_{r-1}) = u$  and  $f(a_1) = x_3$ . then  $f$  is  $T$ -extensible. On the other hand, if  $x_1 = z$  or  $x_2 = z$ , let  $G' = G - \{u, x_1, x_2, x_3\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Using the same above argument, we can prove the assertion.

(A.3).  $d_G(x_1) = k$  and  $d_G(x_2) \leq k-2$ . By (A.2), we can assume that  $d_G(x_3) \leq k-2$ . If  $z \neq x_2, x_3$ , let  $G' = G - \{u, z, x_1, x_2, x_3\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Hence  $e(G') \geq e(G) - k - (k-5) - (k-2) - k - (k-2) + 2 > \frac{1}{2}(k^2 - 8k + 10)$ , which implies  $\text{avedeg}(G') > (k^2 - 8k + 10)/(k-1) > k-7$  and  $|V(T')| \leq k-5$ . By the induction hypothesis,  $T' \subseteq G'$ . Note that  $x_1$  misses at most one vertex in  $V(G')$ . If  $x_1$  misses  $f'(a_2)$ , let  $f(a_1) = u$  and  $f(a_{r-1}) = x_1$ ; if  $x_1$  hits  $f'(a_2)$ , let  $f(a_{r-1}) = u$  and  $f(a_1) = x_1$ . Hence  $f$  is  $T$ -extensible. On the other hand, if  $x_2 = z$  or  $x_3 = z$ , let  $G' = G - \{u, x_1, x_2, x_3\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . By the same above argument, we can prove the assertion.

(A.4). There exists at most one vertex in  $\{x_1, x_2, x_3\}$  with degree at most  $k-1$ . Then there exists a vertex  $u'$  in  $V(G) \setminus \{x_1, x_2, x_3, u\}$  with degree at least  $k-1$ . Otherwise, by  $\delta(G) \leq k-5$ , we have  $\text{avedeg}(G) \leq \frac{k+(k-1)(k-2)+(k-1)+2(k-2)+(k-5)}{k+4} \leq k-2$ , which is a contradiction. Let  $G' = G - \{u, u'\} - \{x_1x_2\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - k - k + 1 - 1 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $\text{avedeg}(G') > (k^2 - 2k - 8)/(k+2) = k-4$  and  $|V(T')| \leq k-2$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2)$  hits  $u$ , let  $f(a_1) = u$ ; if  $f'(a_2)$  misses  $u$ , let  $f(a_2) = u$  and  $f(a_1) = u'$ . Then  $f$  is  $T$ -extensible.

(B).  $d_T(a_1) = 2$  and  $d_T(a_{r-1}) = 2$ . If there exists a vertex  $w$  that hits both

$x_1$  and  $x_3$ , let  $G' = G - \{u, w\} - \{x_1 x_2\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - 2k + 1 - 1 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $avedeg(G') > (k^2 - 2k + 8)/(k + 2) = k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_1$  or  $x_3$ , let  $f(a_1) = w$  and  $f(a_0) = u$ ; if  $f'(a_2) = x_2$  and  $x_1 \notin f'(V(T'))$ , let  $f(a_1) = x_1$  and  $f(a_0) = w$ ; if  $f'(a_2) = x_2$  and  $x_1 \in f'(V(T'))$ , let  $f'(v) = u$ ,  $f(a_1) = x_1$  and  $f(a_0) = w$ . In the above situations,  $f$  is  $T$ -extensible. On the other hand, if there is no vertex hits both  $x_1$  and  $x_3$ , or  $x_2$  and  $x_3$ . then  $d_G(x_1) + d_G(x_3) \leq k$ ,  $d_G(x_2) + d_G(x_3) \leq k$ . Since  $d_G(x_i) \geq \lfloor \frac{k}{2} \rfloor$  and  $k \geq 9$ ,  $d_G(x_i) \leq k - 2$ . Hence Similar to (A.4), there exists a vertex hits  $u$  with degree greater than  $k - 1$  and an embedding of  $T$  into  $G$ .

### 2.4.3 $G[S]$ contains exactly two edges

Without loss of the generality, assume that  $x_1$  hits both  $x_2$  and  $x_3$ . We consider the two cases.

(A).  $d_T(a_1) = 2$ . Let  $G' = G - \{u, x_1\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - 2k > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $avedeg(G') > (k^2 - 2k - 8)/(k + 2) > k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_2$  or  $x_3$  (say  $x_2$ ), let  $f(a_1) = x_1$ ; Moreover, if  $x_3 \notin f'(V(T'))$ , let  $f(a_0) = x_3$ ; if  $x_3 \in f'(V(T'))$  and  $f'(v) = x_3$ , let  $f'(v) = u$ ,  $f(a_1) = x_1$ , and  $f(a_0) = x_3$ . Hence,  $f$  is  $T$ -extensible. If  $f'(a_2) \neq x_2, x_3$ , then it is easy to find an embedding from  $T$  to  $G$ .

(B).  $d_T(a_1) \geq 3$ .

(a).  $d_G(x_1) \geq k - 1$ . If  $z \neq x_2, x_3$ , let  $G' = G - \{u, z, x_1\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - k - (k - 5) - k + 1 > \frac{1}{2}(k^2 - 4k + 4)$ , which implies  $avedeg(G') > (k^2 - 4k + 4)/(k + 1) > k - 5$  and  $|V(T')| \leq k - 3$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_2$  or  $x_3$  (say  $x_2$ ), let  $f(a_1) = x_1$ . Moreover, if  $x_3 \notin f'(V(T'))$ , let  $f(a_0) = x_3$ ; if  $x_3 \in f'(V(T'))$  and  $f'(v) = x_3$ , let  $f'(v) = u$ ,  $f(a_1) = x_1$  and  $f(a_3) = v$ , because  $u$  hits all neighbours of  $f'(v)$ . Hence  $f$  is  $T$ -extensible. If  $f'(a_2) \neq x_2, x_3$ , it is easy to find an embedding from  $T$  to  $G$ . On the other hand, if  $z = x_2$  or  $x_3$  (say  $x_2$ ), let  $G' = G - \{u, x_1, x_2\}$ , by the same argument as (a), the assertion holds.

(b).  $d_G(x_1) \leq k - 2$ ,  $d_G(x_2) = k$  or  $d_G(x_3) = k$  (say  $d_G(x_2) = k$ ). Then there exists a vertex  $y \in V(G) \setminus \{u, x_1, x_2, x_3\}$  such that  $x_2$  misses  $y$ . So  $x_2$  misses  $u, x_3$  and  $y$  and  $u$  misses  $x_3$ . By Case 2.4.2, we can assume  $y$  hits  $x_3$ . Further, by (a), we can assume  $d_G(y) \leq k - 2$ . If  $z \neq x_1, y$ , let  $G' = G - \{u, z, x_2, x_3, y\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - k - (k - 5) - k - k -$

$(k-2)+3 > \frac{1}{2}(k^2-8k+12)$ , which implies  $\text{avedeg}(G') > (k^2-8k+12)/(k-1) > k-7$  and  $|V(T')| \leq k-5$ . By the induction hypothesis,  $T' \subseteq G'$ . Further, if  $f'(a_2) = x_1$ , let  $f(a_1) = x_2$  and  $f(a_{r-1}) = u$ ; if  $f'(a_{r-2}) = x_1$ , let  $f(a_{r-1}) = x_2$  and  $f(a_1) = u$ . Hence  $f$  is  $T$ -extensible. On the other hand, if  $z = y$ , let  $G' = G - \{u, x_2, x_3, y\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ ; if  $z = x_1$ , let  $G' = G - \{u, z, x_2, x_3, y\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then by the same argument, it is easy to prove that the assertion holds.

(c).  $d_G(x_1) \leq k-2$ ,  $d_G(x_2) = k-1$  and  $d_G(x_3) = k-1$ . Let  $G' = G - \{u, x_2, x_3\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - k - (k-1) - (k-1) > \frac{1}{2}(k^2 - 4k - 4)$ , which implies  $\text{avedeg}(G') > (k^2 - 4k - 4)/(k+1) > k-5$  and  $|V(T')| \leq k-3$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_1$ , let  $f(a_1) = x_2$ , which  $f$  is  $T$ -extensible. If  $f'(a_2) \neq x_1$ , it is easy to find an embedding from  $T$  to  $G$ .

(d).  $d_G(x_1) \leq k-2$ , and  $d_G(x_2) \leq k-2$  or  $d_G(x_3) \leq k-2$  (say  $d_G(x_2) \leq k-2$ ), hence  $d_G(x_3) \leq k-1$  by (b). Then there exists a vertex  $u' \in V(G) \setminus \{x_1, x_2, x_3, u\}$  of degree at least  $k-1$ , otherwise  $2e(G) \leq (k-1)(k-2) + (k-5) + k + 2(k-2) + (k-1) \leq (k+4)(k-2)$  which is impossible. Let  $G' = G - \{u, u', x_1\}$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - 2k - (k-2) + 1 > \frac{1}{2}(k^2 - 4k - 2)$ , which implies  $\text{avedeg}(G') > (k^2 - 4k - 2)/(k+1) > k-5$  and  $|V(T')| \leq k-3$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2)$  hits  $u$ , let  $f(a_1) = u$ ; if  $f'(a_2) = x_2$  or  $x_3$  (say  $x_2$ ), let  $f'(a_2) = u$  and  $f(a_1) = u'$  since  $u$  hits all the neighbours of  $f'(a_2)$ . Then  $f$  is  $T$ -extensible.

#### 2.4.4 $G[S]$ contains exactly three edges

(A).  $d_T(a_1) = 2$ . If there exists an  $1 \leq i \leq 3$  (say  $i = 1$ ) such that  $d_G(x_1) \leq k-1$ , let  $G' = G - \{u, x_1\} - \{x_2x_3\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - k - (k-1) - 1 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $\text{avedeg}(G') > (k^2 - 2k - 8)/(k+2) > k-4$  and  $|V(T')| \leq k-2$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_2$  or  $x_3$  (say  $x_2$ ), let  $f(a_1) = x_1$ . Moreover, if  $x_3 \notin f'(V(T'))$ , let  $f(a_0) = x_3$ ; and if  $x_3 \in f'(V(T'))$  and  $f'(v) = x_3$ , let  $f'(v) = u$ ,  $f(a_1) = x_1$ ,  $f(a_0) = x_3$  and  $f(a_3) = v$ . Hence  $f$  is  $T$ -extensible. On the other hand, if  $d_G(x_1) = d_G(x_2) = d_G(x_3) = k$ , let  $G' = G - \{u, x_1\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - 2k > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $\text{avedeg}(G') > (k^2 - 2k - 8)/(k+2) = k-4$  and  $|V(T')| \leq k-2$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_2$  or  $x_3$ , let  $f(a_1) = x_1$ ; if  $f'(a_2) \neq x_2, x_3$ , let  $f(a_1) = u$ . Hence  $f$  is  $T$ -extensible.

(B).  $d_T(a_1) \geq 3$ . If there exists an  $1 \leq i \leq 3$  (say  $i = 1$ ) such that  $d_G(x_1) \geq k - 1$ , Let  $G' = G - \{u, z, x_1\} - \{x_2x_3\}$ . By the same argument as Case 2.4.3.(B).(a)., the assertion holds. The rest is similar as Case 2.4.3.(B).(d).

## 2.5 $\Delta(G) = k - 1$

Since  $\Delta(G) = k - 1$  and  $\delta(G) \geq k - 5$ , there exist at least four vertices of degree  $k - 1$ . Otherwise  $2 \leq 3(k - 1) + k(k - 2) + (k - 5) = (k - 2)(k + 4)$ , which is a contradiction. Let  $u_i$  be vertex of  $d_G(u_i) = k - 1$  missing four vertices of  $S_i = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}\}$  for  $i = 1, 2, 3, 4$ . If there exists a vertex  $u_i$  with  $1 \leq i \leq 4$  such that  $G[S_i]$  contains at most one edge. let  $G' = G - \{u_i\} - E(G[S_i])$  and  $T' = T - \{a_0\}$ . Then  $e(G') \geq e(G) - (k - 1) - 1 > \frac{1}{2}(k^2 - 8)$ , which implies  $avedeg(G') > (k^2 - 8)/(k + 3) > k - 3$  and  $|V(T')| \leq k - 1$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $u_i$  hits  $f'(a_1)$ , let  $f(a_0) = u_i$ , and if  $u_i$  misses  $f'(a_1)$ , let  $f'(a_1) = u_i$ . Then  $f$  is  $T$ -extensible. Hence we assume that  $G[S_i]$  contains at least two edges for  $i = 1, 2, 3, 4$ .

### 2.5.1 $d_T(a_1) \geq 3, d_T(a_{r-1}) \geq 2$

(A).  $G[u_1, u_2, u_3, u_4]$  contains at least one edge, say  $u_1$  hits  $u_2$ . If  $z \notin S_1 = \{x_{11}, x_{12}, x_{13}, x_{14}\}$ , let  $G' = G - \{u_1, u_2, z\} - E(G[S_1])$  and  $T' = T - \{a_1, b_1, \dots, b_s\}$ . Then  $e(G') \geq e(G) - 2(k - 1) - (k - 5) + 1 - 6 > \frac{1}{2}(k^2 - 4k - 4)$ , which implies  $avedeg(G') > (k^2 - 4k - 4)/(k + 1) > k - 5$  and  $|V(T')| \leq k - 3$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $u_1$  hits  $f'(a_2)$ , let  $f(a_1) = u_1$ ; and if  $u_1$  misses  $f'(a_2)$ , let  $f'(a_2) = u_1$  and  $f(a_1) = u_2$ . Since  $u_1$  hits all the neighbours of  $f'(a_2)$ ,  $f$  is  $T$ -extensible. On the other hand, if  $z \in S_1 = \{x_{11}, x_{12}, x_{13}, x_{14}\}$ , say  $z = x_{11}$ . Let  $G' = G - \{u_1, u_2, z\} - E(G[x_{12}, x_{13}, x_{14}])$ . By the same argument, the assertion holds.

(B).  $G[u_1, u_2, u_3, u_4]$  contains no edges.

(B.1). If there exist two vertices, say  $u_1$  and  $u_2$ , in  $\{u_1, u_2, u_3, u_4\}$  such that  $u_1$  misses  $y_1$  and  $u_2$  misses  $y_2$ , where  $y_1 \neq y_2$  and  $y_1, y_2 \notin \{u_1, \dots, u_4\}$ . Let  $G' = G - \{u_1, u_2, u_3, u_4\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - 4(k - 1) > \frac{1}{2}(k^2 - 6k)$ , which implies  $avedeg(G') > (k^2 - 6k)/k = k - 6$  and  $|V(T')| \leq k - 4$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2) = y_1$ , let  $f(a_1) = u_2$  and  $f(a_{r-1}) = u_1$ ; if  $f'(a_2) = y_2$ , let  $f(a_1) = u_1$  and  $f(a_{r-1}) = u_2$ . Moreover, if  $f'(a_{r-2}) = y_1$ , let  $f(a_1) = u_1$  and  $f(a_{r-1}) = u_2$ ; and if  $f'(a_{r-2}) = y_2$ , let  $f(a_1) = u_2$  and  $f(a_{r-1}) = u_1$ . Therefore,  $f$  is  $T$ -extensible.

(B.2). There exist a vertex  $y \notin \{u_1, \dots, u_4\}$  such that  $y$  misses  $u_1, \dots, u_4$ . Then  $G[u_1, \dots, u_4, y]$  contains no edges.

(a).  $d_T(a_{r-1}) = 2$ . Then there exists a vertex  $w$  hits  $\{u_1, u_2, u_3, u_4\}$  and  $y$ . Let  $G' = G - \{u_1, w\}$  and  $T' = T - \{a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 2(k-1) + 1 > \frac{1}{2}(k^2 - 2k - 2)$ , which implies  $avedeg(G') > (k^2 - 2k - 2)/(k+2) > k-4$  and  $|V(T')| \leq k-2$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_{r-2}) = u_2, u_3, u_4$  or  $y$ , let  $f(a_{r-1}) = w$  and  $f(a_r) = u_1$ ; and if  $f'(a_{r-2}) \neq u_2, u_3, u_4, y$ , let  $f(a_{r-1}) = u_1$  and  $f(a_r) = w$ . Therefore  $f$  is  $T$ -extensible.

(b).  $d_T(a_{r-1}) \geq 3$ . If  $z \neq y$ , let  $G' = G - \{u_1, u_2, u_3, u_4, y, z\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . Then  $e(G') \geq e(G) - 4(k-1) - (k-1) - (k-5) + 4 > \frac{1}{2}(k^2 - 10k + 20)$ , which implies  $avedeg(G') > (k^2 - 10k + 20)/(k-2) > k-8$  and  $|V(T')| \leq k-6$ . By the induction hypothesis,  $T' \subseteq G'$ . Let  $f(a_1) = u_1$  and  $f(a_{r-1}) = u_2$ . Then  $f$  is  $T$ -extensible. On the other hand, if  $z = y$ , let  $G' = G - \{u_1, u_2, u_3, u_4, z\}$  and  $T' = T - \{a_1, b_1, \dots, b_s, a_{r-1}, c_1, \dots, c_t\}$ . By the same argument, the assertion holds.

### 2.5.2 $d_T(a_1) = 2, d_T(a_{r-1}) = 2$ .

(A). There exists a  $1 \leq i \leq 4$ , say  $i = 1$ , such that  $G[S_1]$  contains two or three edges. If  $u_1$  hits one vertex, say  $u_2$ , of three vertices  $u_2, u_3, u_4$ . Let  $G' = G - \{u_1, u_2\} - E(G[S_1])$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $avedeg(G') > (k^2 - 2k - 8)/(k+2) = k-4$  and  $|V(T')| \leq k-2$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $u_1$  hits  $f'(a_2)$ , let  $f(a_1) = u_1$ ; and if  $u_1$  misses  $f'(a_2)$ , let  $f'(a_2) = u_1$  and  $f(a_1) = u_2$ . Since  $u_1$  hits all the neighbours of  $f'(a_2)$ ,  $f$  is  $T$ -extensible. Therefore, we assume that  $u_1$  misses  $u_j$  for  $j = 2, 3, 4$ . Then  $u_1$  misses  $x_{11} = u_2, x_{12} = u_3, x_{13} = u_4, x_{14}$  and  $G[u_2, u_3, u_4, x_{14}]$  contains two or three edges.

(A.1).  $x_{14}$  hits one vertex, say  $u_2$ , of three vertices  $u_2, u_3, u_4$ . Let  $G' = G - \{u_1, u_2, u_3, u_4\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 4(k-1) > \frac{1}{2}(k^2 - 6k)$ , which implies  $avedeg(G') > (k^2 - 6k)/k = k-6$  and  $|V(T')| \leq k-4$ . By the induction hypothesis,  $T' \subseteq G'$ . Since  $G[u_2, u_3, u_4, x_{14}]$  contains two or three edges, there exists a vertex, say  $u_3$ , of two vertices  $u_3, u_4$  misses at most one vertex, say  $y_1$ , in  $V(G) \setminus \{u_1, u_2, u_4, x_{14}\}$ . Hence if  $f'(a_2) = x_{14}$  or  $y_1$ , and  $f'(a_{r-2}) = y_1$  or  $x_{14}$ , let  $f(a_1) = u_2$  or  $u_1$  and  $f(a_{r-1}) = u_1$  or  $u_2$ , then  $f$  is  $T$ -extensible. For the rest cases, it is easy to find an embedding from  $T$  to  $G$ .

(A.2).  $x_{14}$  misses three vertices  $u_2, u_3, u_4$ . Then  $G[u_2, u_3, u_4]$  contains two or three

edges. We can assume that  $u_2$  hits  $u_3$  and  $u_4$ . If  $u_3$  misses  $u_4$ ,  $u_3$  misses at most one vertex, says  $y_1$ , in  $V(G) \setminus \{u_1, u_2, u_4, x_{14}\}$ . Then let  $G' = G - \{u_1, x_{14}, u_3, u_4\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . By the similar argument as Case (A.1), the assertion holds. Hence we can assume that  $u_3$  hits  $u_4$  and  $u_3$  misses  $z_1, z_2, u_1, x_{14}$ . Let  $G' = G - \{u_1, x_{14}, u_3, u_4\} - \{z_1 z_2\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 4(k-1) + 1 - 1 > \frac{1}{2}(k^2 - 6k)$ , which implies  $avedeg(G') > (k^2 - 6k)/k = k - 6$  and  $|V(T')| \leq k - 4$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2) = z_1$  or  $z_2$ , and  $f'(a_{r-2}) = z_2$  or  $z_1$ , let  $f'(a_2) = u_3$ ,  $f(a_1) = u_4$ ,  $f(a_{r-1}) = u_1$ . Therefore  $f$  is  $T$ -extensible. If  $f'(a_2) = z_1$  or  $z_2$ , and  $f'(a_{r-2}) = u_2$ , let  $f(a_1) = u_1$ ,  $f(a_{r-1}) = u_4$ . Therefore  $f$  is  $T$ -extensible. For the rest cases, it is easy to find an embedding from  $T$  to  $G$ .

(B). There exists a  $1 \leq i \leq 4$ , say  $i = 1$ , such that  $G[S_1]$  contains exactly four edges.

(B.1). There exists a vertex, say  $x_{11}$ , of degree 3 in  $G[S_1]$  and  $|E(G[S_1])| \leq 5$ . Then  $x_{11}$  hits  $x_{12}, x_{13}$  and  $x_{14}$ . Let  $G' = G - \{u_1, x_{11}\} - \{E(G[x_{12}, x_{13}, x_{14}])\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - 2(k-1) - 2 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $avedeg(G') > (k^2 - 2k - 8)/(k+2) > k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $u_1$  hits  $f'(a_2)$ , let  $f(a_1) = u_1$ , which implies that  $f$  is  $T$ -extensible. If  $u_1$  misses  $f'(a_2)$  and  $f'(a_2) = x_{12}$ , let  $f(a_1) = x_{11}$ . Moreover, if  $x_{13}$  or  $x_{14} \notin f'(V(T'))$ , then let  $f(a_0) = x_{13}$  or  $x_{14}$ . Then  $f$  is  $T$ -extensible. If  $x_{13}$  and  $x_{14} \in f'(V(T'))$ ,  $f'(w) = x_{13}$  or  $x_{14}$ , let  $f'(w) = u_1$ ,  $f(a_0) = x_{13}$  or  $x_{14}$ . Then  $f$  is  $T$ -extensible. For the rest cases, it is easy to find an embedding from  $T$  to  $G$ .

(B.2). The degree of every vertex in  $G[S_1]$  is two. We assume that  $x_{11}$  hits  $x_{12}, x_{13}$  hits  $x_{13}, x_{13}$  hits  $x_{14}, x_{14}$  hits  $x_{11}$ .

(a).  $u_1$  hits all vertices of  $\{u_2, u_3, u_4\}$ .

(a.1). There exists a vertex  $u_i$ , say  $u_2$ , in  $\{u_2, u_3, u_4\}$  which misses  $x_{11}, x_{12}, x_{13}$  and  $x_{14}$ . Let  $G' = G - \{u_1, u_2, x_{11}, x_{12}\} - \{x_{13}x_{14}\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 4(k-1) + 1 > \frac{1}{2}(k^2 - 6k + 2)$ , which implies  $avedeg(G') > (k^2 - 6k + 2)/k > k - 6$  and  $|V(T')| \leq k - 4$ . By the induction hypothesis,  $T' \subseteq G'$ . If  $f'(a_2) = x_{13}, f'(a_{r-2}) = x_{14}$ , let  $f(a_1) = x_{12}, f(a_0) = x_{11}, f(a_{r-2}) = u_1, f(a_{r-1}) = u_2$ . Hence  $f$  is  $T$ -extensible. For the rest cases, similarly, it is easy to find an embedding from  $T$  to  $G$ .

(a.2). There exists a vertex, say  $u_2$ , in  $\{u_2, u_3, u_4\}$  such that it hits at least two vertices of  $\{x_{11}, x_{12}, x_{13}, x_{14}\}$ , say  $u_2$  hits  $x_{11}$  and  $x_{13}$ , or  $u_2$  hits  $x_{11}$  and  $x_{12}$ .

If  $u_2$  hits  $x_{11}$  and  $x_{13}$ , let  $G' = G - \{u_1, u_2\} - \{x_{11}x_{12}, x_{12}x_{13}, x_{13}x_{14}\}$  and  $T' =$

$T - \{a_0, a_1, a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $\text{avedeg}(G') > k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2) = x_{11}$  or  $x_{13}$ , let  $f(a_1) = u_2$ ; if  $f'(a_2) = x_{12}$ , let  $f(a_2) = u_1$  and  $f(a_1) = u_3$ ; if  $f'(a_2) = x_{14}$  and  $x_{13} \notin f'(V(T'))$ , let  $f(a_1) = x_{13}$  and  $f(a_0) = u_2$ ; if  $f'(a_2) = x_{14}$  and  $x_{13} \in f'(V(T'))$ , let  $f(v) = u_1, f(a_1) = x_{13}, f(a_0) = u_2$ , because there is a vertex  $v, f'(v) = x_{13}$  and  $u_1$  hits all the neighbours of  $f'(v)$ . Therefore  $f$  is  $T$ -extensible.

If  $u_2$  hits  $x_{11}$  and  $x_{12}$ , let  $G' = G - \{u_1, u_2\} - \{x_{12}x_{13}, x_{13}x_{14}, x_{11}x_{14}\}$  and  $T' = T - \{a_0, a_1\}$ . Then  $e(G') \geq e(G) - 2(k-1) + 1 - 3 > \frac{1}{2}(k^2 - 2k - 8)$ , which implies  $\text{avedeg}(G') > k - 4$  and  $|V(T')| \leq k - 2$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2) = x_{11}$  or  $x_{12}$ , let  $f(a_1) = u_2$ ; if  $f'(a_2) = x_{13}$  or  $x_{14}$ , let  $f(a_2) = u_1, f(a_1) = u_2$ , because  $u_1$  hits all the neighbours of  $f'(a_2)$ . Therefore  $f$  is  $T$ -extensible.

(a.3).  $u_i$  hits exactly one vertex of  $\{x_{11}, x_{12}, x_{13}, x_{14}\}$  for  $i = 2, 3, 4$ .

(i). There exist two vertices of  $\{u_2, u_3, u_4\}$  such that they hit the same vertex in  $\{x_{11}, x_{12}, x_{13}, x_{14}\}$ , says both  $u_2$  and  $u_3$  hit  $x_{14}$ .

If  $u_2$  and  $u_3$  misses the same vertices, say,  $\{x_{11}, x_{12}, x_{13}, y\}, u_3$ , then  $u_2$  hits  $u_3$ . Further, if  $G[x_{11}, x_{12}, x_{13}, y]$  contains at most three edges or has a vertex of degree 3, the assertion follows from Case 2.5.2.(A) or Case 2.5.2.(B.1). Therefore we can assume that  $y$  hits both  $x_{11}$  and  $x_{13}$ . Let  $G' = G - \{u_2, u_3, x_{11}, x_{12}\} - \{x_{13}y\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . The assertion follows from Case 2.5.2. (B.2).(a.1).

If  $u_2$  misses  $\{x_{11}, x_{12}, x_{13}, y_1\}$  and  $u_3$  misses  $\{x_{11}, x_{12}, x_{13}, y_2\}$  with  $y_1 \neq y_2$ , let  $G' = G - \{u_1, u_2, u_3, x_{14}\} - \{x_{11}x_{12}, x_{12}x_{13}\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 4(k-1) + 4 - 2 > \frac{1}{2}(k^2 - 6k + 4)$ , which implies  $\text{avedeg}(G') > k - 6$  and  $|V(T')| \leq k - 4$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2) = x_{11}$  or  $x_{13}$ , let  $f(a_1) = x_{14}, f(a_0) = u_3$  or  $u_2$ . if  $f'(a_2) = x_{12}$ , let  $f(a_2) = u_1, f(a_1) = u_3$  or  $u_2$ . if  $f'(a_2) = y_1$  or  $y_2$ , let  $f(a_1) = u_3$  or  $u_2$ . which implies  $f$  is  $T$ -extensible. For the rest cases, it is easy to find an embedding from  $T$  to  $G$ .

(ii).  $\{u_2, u_3, u_4\}$  hits the different vertices of  $\{x_{11}, x_{12}, x_{13}, x_{14}\}$ . Without loss of generality, we assume that  $u_2$  hits  $x_{11}$  and  $u_3$  hits  $x_{13}, u_2$  misses  $y_1$  and  $u_3$  misses  $y_2$ . Let  $G' = G - \{u_1, u_2, u_3, x_{13}\} - \{x_{11}x_{12}, x_{11}x_{14}\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 4(k-1) + 3 + 0 - 2 > \frac{1}{2}(k^2 - 6k + 2)$ , which implies  $\text{avedeg}(G') > k - 6$  and  $|V(T')| \leq k - 4$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2) = x_{12}$  or  $x_{14}$ , let  $f(a_1) = x_{13}$  and  $f(a_0) = u_3$ , or  $f(a_2) = u_1$  and  $f(a_1) = u_2$ , if  $f'(a_2) = y_1$  or  $y_2$ , let  $f(a_1) = u_1$ , if  $f'(a_2) = x_{11}$ , let  $f(a_1) = u_2$ , Therefore  $f$  is  $T$ -extensible. For the rest cases, by the same argument, it is easy to find an embedding from  $T$  to  $G$ .

(b).  $u_1$  hits one or two vertices of  $\{u_2, u_3, u_4\}$ . Without loss of the generality, we assume that  $u_1$  hits  $u_2$  and  $u_1$  misses  $u_4$ . Then  $u_4 \in \{x_{11}, x_{12}, x_{13}, x_{14}\}$ , say  $u_4 = x_{14}$ ,  $u_4$  misses  $u_1, x_{12}, z_1, z_2$ .

If  $u_2 \neq z_1, z_2$ , let  $G' = G - \{u_1, u_2, u_4, x_{12}\} - \{z_1 z_2\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . Then  $e(G') \geq e(G) - 4(k-1) + 1 - 1 > \frac{1}{2}(k^2 - 6k)$ , which implies  $\text{avedeg}(G') > k - 6$  and  $|V(T')| \leq k - 4$ . By the induction hypothesis,  $T' \subseteq G'$ . Hence if  $f'(a_2) = x_{11}$  and  $f'(a_{r-2}) = x_{13}$ , let  $f(a_1) = u_4, f(a_{r-2}) = u_1$  and  $f(a_{r-1}) = u_2$ . Therefore  $f$  is T-extensible. For the rest cases, it is easy to find an embedding from  $T$  to  $G$ . If  $u_2 = z_1$  or  $z_2$ , say  $u_2 = z_1$ , let  $G' = G - \{u_1, u_2, u_4, x_{12}\}$  and  $T' = T - \{a_0, a_1, a_{r-1}, a_r\}$ . This situation is much easier than the above case.

(c).  $u_1$  misses all vertices of  $\{u_2, u_3, u_4\}$ . Without loss of generality, we assume  $u_2 = x_{11}, u_3 = x_{12}, u_4 = x_{13}$ . Let  $u_2$  miss  $\{u_1, x_{13}, y_1, y_2\}$ . If  $G[u_1, x_{13}, y_1, y_2]$  contains two, or three edges, or a vertex of degree 3, the assertion follows from Case 2.5.2 (A). and Case 2.5.2 (B.1). Hence we assume that  $u_1$  hits  $y_1$ ,  $y_1$  hits  $u_4 = x_{13}$ ,  $u_4$  hits  $y_2$  and  $y_2$  hits  $u_1$ . Hence the assertion follows from Case 2.5.2. (B.2). (a) and Case 2.5.2. (B.2).(b).

(C). There exists a  $1 \leq i \leq 4$ , say  $i = 1$ , such that  $G[x_{11}, x_{12}, x_{13}, x_{14}]$  contains five edges. Then we assume that  $x_{11}$  hits  $x_{12}, x_{13}$  and  $x_{14}$ . Let  $G' = G - \{u_1, x_{11}\} - \{E(G[x_{12}, x_{13}, x_{14}])\}$  and  $T' = T - \{a_0, a_1\}$ . The assertion follows from the proof of Case 2.5.2 (B.1).

(D). There exists a  $1 \leq i \leq 4$ , say  $i = 1$ , such that  $G[x_{11}, x_{12}, x_{13}, x_{14}]$  contains six edges. If  $d_G(x_{i1}) \leq k - 2$ , similar as Case 2.5.2 (B.1), we can prove the assertion. So we can assume  $d_G(x_{i1}) = d_G(x_{i2}) = d_G(x_{i3}) = d_G(x_{i4}) = k - 1$ , we can also assume if  $d_G(x) = k - 1$ , and  $x$  misses  $y$  then  $d_G(y) = k - 1$ , furthermore we can assume  $x$  hits all of the vertices whose degree is less than  $k - 1$ . let  $G' = G - \{u_1, z\}$ ,  $z$  hits all of  $\{x_1, x_2, x_3, x_4\}$ ,  $T' = T - \{a_0, a_1\}$ . So  $e(G') \geq e(G) - (k-1) - (k-5) + 1 > \frac{1}{2}(k+4)(k-2) - (k-1) - (k-5) + 1 = \frac{1}{2}(k^2 - 2k + 6)$ .  $\text{avedeg}(G') > (k^2 - 2k + 6)/(k+2) > k - 4$  and  $|V(T')| \leq k - 2$ . By the induction assumption,  $T' \subseteq G'$ . If  $f'(a_2)$  hits  $u_1$ , then  $f(a_1) = u_1, f(a_0) = z$ . If  $f'(a_2)$  misses  $u_1$ , then  $f(a_0) = u_1, f(a_1) = z$ .  $f$  is T-extensible.

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